An Exit Hole method for Verified Solution of IVPs for ODEs using Linear Programming for the Search of Tight Bounds

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Abstract

In his survey [5], NEDIALKOV stated that “Although high-order Taylor series may be reasonably efficient for mildly stiff ODEs, we do not have an interval method suitable for stiff ODEs.” This paper is an attempt to find such a method, based on building a positively invariant set in extended state space. A positively invariant set is treated as geometric generalization of differential inequalities. We construct a positively invariant set from simpler sets which are not positively invariant, but have exit hole instead. The exit holes of simpler sets are suppressed during the construction. This paper considers only sets which are polytopes. Linear interval forms are used to evaluate a projection of ODE velocity vector to the normals of the polytop facets. This permits the use of Linear Programming for the search of tighter positively invariant set. The Exit Hole method is illustrated by stiff Van der Pol ODE.

1 Introduction

In this paper we consider an IVP defined as in [5]

$$\dot{x}(t) = f(x(t), t),\ x(t_0) = x_0,\ x \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{1}$$

We convert this problem in an autonomous IVP

$$\dot{y}(s) = F(y(s)) = \begin{pmatrix} f(x(s), t+s) \\ 1 \end{pmatrix},\ y(0) = y_0 = \begin{pmatrix} x_0 \\ t_0 \end{pmatrix},\ y \in \mathbb{R}^{n+1},\ s \in \mathbb{R}. \tag{2}$$

The initial condition and initial time of the may be uncertain, that is $y_0 \in \boldsymbol{y}_0$. We want to compute rigorous bounds on the true solution of (1) which enclose both uncertainty in inital conditions and roundoff and truncation errors of numerical computation. We build *positively invariant set* $\Omega \subseteq \mathbb{R}^{n+1}$ of (2) for this purpose. A positively invariant set contains a positive semi-orbit starting from every its point.

Positively invariant set is a global property of the set. We use a local notion of an *exit point* of ODE (2) from Ω which is defined by structure of Ω in the neighbourhood of the exit point only. All exit points lie on the boundary $\partial\Omega$. The set of all exit points of ODE from a set is an *exit hole* of ODE from a set. If the exit hole of ODE from a set is empty, then this set is the postively invariant set of the ODE.

We build a positively invariant set from simpler sets which are not positively invariant sets yet. The build operations are union and intersection. The exit hole of the result of each operation is derived from exit holes of operand sets. The simplest sets are those with smooth boundary.

This paper considers only simplest sets which are closed half-spaces with hyperplane boundary of dimension n. $\Omega = \{y | c^T y \geqslant \boldsymbol{a}\}$ with inward-pointing normal $c \in \mathbb{R}^{n+1}$, $c \neq 0$. Their exit holes are determined by the projection of ODE velocity vector on the surface and the normal $c^T F(y_e) <= 0$. The $F(y)$ is non-linear function defined by expression $\mathsf{F}(y)$. We use interval methods to enclose it. If we enclose it by linear function with interval coefficients (by mean-value evaluation or by slope evaluation), we can use linear programming for the search of tighter invariant sets.

The intersection of half-spaces is a convex polytop of dimension $n+1$, where n is the number of ODEs in the system. This paper considers only polytops which are $(n+1)$-dimensional trapezoids. Its parallel surfaces of the trapezoid are n-dimensional boxes and they are orthogonal either to time axis or to other integration variable axis. The exit hole is one of the parallel surfaces of the trapezoid. The linear programming is used to search for a trapezoid with smaller exit hole.

Each trapezoid represents a step. The union of a sequence of trapezoids is built. The initial point belongs to the first trapezoid. The exit hole of a previous trapezoid is contained in the next trapezoids. The exit hole of the last trapezoid is terminated by a half-space with normal directed to positive time axis. The resulting union is a positively invariant set in extended state space. It is an enclosure of the solution of IVP with the initial point is in the set.

The rigorous solution of stiff problems was considered by NEUMAIER [1] and KÜHN [2]. The Neumaier's method was analized by YU [3] and implemented by YUK [4]. Their papers enclosed the solution by balls of some norm in R^n, which is fixed along time. The general Exit Hole method announces about arbitrary smooth boundaries and abitrary polytopes. However, the Exit Hole method in this paper considers only trapezoids. So each step enclosure in this paper can be represented by any of these methods. The difference is that Exit Hole method uses separate interval mean-value evaluation for each surface of trapezoid, while these two methods use the same mean-value evaluation for whole body of trapezoid. So Exit Hole needs more work, but for nonlinear ODEs with wide enclosure it may have less overestimation. The ValEncIA-IVP solver [6] uses mean-value evaluation for body of trapezoid, but it compensates overestimation by spliting the body of trapezoid adaptively. This compensates overestimation.

Rauh, Auer, Minisini, Hofer in [7] suggested Exponential State Enclosures. So dissipative components of state are approximated as $x_0[e^{\underline{\lambda}t}, e^{\overline{\lambda}t}]$ instead of $x_0[1+\underline{\lambda}t, 1+\overline{\lambda}t]$. This prevents the enclosure from being negative. The Exit Hole method gives wider enclosure $x_0[\max(1+\underline{\lambda}t, 0), \frac{1}{1-\lambda t}]$.

Gennat, Tibken in [8] used Müller's theorem from the theory of differential inequalities. Also they used linear Lyapunov-like function - linear surface in R^n which moves along time. They used exact bounds for velocity vectors on each surface for their example. The difference of Exit Hole method is that it can use trapezoids with integration variable other than time. The Van der Pol integration obtained benefits from this feature.

Section 2 contains definition of exit hole and basic facts about it. Section 3 discusses exit holes of half-spaces. It also formulates a problem of optimal bound for scalar ODEs. This formulation helps to understand, but it is too hard to solve it for non-linear ODEs. Section 4 explains how interval evaluation gives a simpler estimation of the exit hole condition. Section 5 tells about exit hole of intersection of sets. It presents trapezoid - a polytope used in this paper, and tells how to search for a tighter trapezoid using linear programming. Section 6 tells about exit hole of union of a sequence of sets. Section 7 contains results of simulationg Van der Pol ODE. Section 8 is the conclusion

2 Exit Hole

Example IVPs We shall illustrate definitions and theorems at four example IVPs.

Dahlquist

$$y = \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^2,\ \dot{y} = F(y) = \begin{pmatrix} -x \\ 1 \end{pmatrix},\ y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

Nonsmooth

$$y = \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^2,\ \dot{y} = F(y) = \begin{pmatrix} \min(x, 1-x) \\ 1 \end{pmatrix},\ y_0 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \tag{4}$$

Curtiss-Hirschfelder

$$y = \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^2,\ \dot{y} = F(y) = \begin{pmatrix} 50(\cos t - x) \\ 1 \end{pmatrix},\ y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

VanDerPol

$$y = \begin{pmatrix} x \\ p \\ t \end{pmatrix} \in \mathbb{R}^3,\ \dot{y} = F(y) = \begin{pmatrix} p \\ \mu(1-x^2)p - x \\ 1 \end{pmatrix},\ y_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},\ \mu = 500 \tag{6}$$

We assume $F(y)$ that is locally Lipschitz continuous in $\mathbb{R}^{n+1}$. For all $y \in \mathbb{R}^{n+1}$ there exists a neighbourhood $U(y) \subseteq \mathbb{R}^{n+1}$ and a Lipschitz constant $L(y)$ such that $y_1 \in U(y), y_2 \in U(y) \Rightarrow \|F(y_1) - F(y_2)\| \leqslant L(y)\|y_1 - y_2\|$.

All four examples are locally Lipschitz continuous, though Nonsmooth example is not differentiable at points $(\frac{1}{2}, t)$ and VanDerPol example is not Lipschitz continuous at the whole domain.

The Picard – Lindelöf Theorem guarantees that for any extended state $y_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ for some $\delta > 0$ there is unique solution $y(s) = (x(s), t_0 + s)$ of ODE (2) in interval $s \in [-\delta, +\delta]$. Moreover, the solution can be prolonged towards either $s \to +\infty$ or $\|x(s)\| \to \infty$. We denote the general solution of (2) stating from y_0 by $y(s; y_0)$.

The general solution of Dahlqust ODE is

$$y(s; y_0) = \begin{pmatrix} x_0 e^{-s} \\ t_0 + s \end{pmatrix}.$$

Definition. The set $S \subseteq \mathbb{R}^{n+1}$ is said to be *positively invariant* if for any point $y_0 = (x_0, t_0) \in S$ the positive semi-orbit from y_0 is contained in S: $\forall s \geqslant 0 \ y(s; y_0) \in S$.

The notion positively invariant set is a global property of the set. We shall try to express it in local terms.

Definition. The point $y_e = (x_e, t_e) \in \mathbb{R}^{n+1}$ is an *exit point* of ODE(2) from a set $\Omega \subseteq \mathbb{R}^{n+1}$ if $y_e \in \overline{\Omega}$ and no starting segment of positive semi-orbit is contained in Ω: $\forall h > 0 \ \exists s \in [0, h[\ y(s; y_e) \notin \Omega$. The exits point from a set is necessariy belongs to the border of the set $y_e \in \partial\Omega$.

Definition. The *exit hole* $E\Omega$ is a set of all exit points of ODE from a set Ω. It follows from the definition that $E\emptyset = \emptyset$ and $E\mathbb{R}^{n+1} = \emptyset$.

Let $\Omega = \{(x,t) |\ x \geqslant a,\ t \in \mathbb{R}\}$. Then
for Dahlquist ODE

$$E\Omega = \begin{cases} \emptyset, & \text{for } a \leqslant 0 \\ \{(a,t)|\ t \in \mathbb{R}\}, & \text{for } a > 0 \end{cases}$$

for Nonsmooth ODE

$$E\Omega = \begin{cases} \emptyset, & \text{for } 0 \leqslant a \leqslant 1 \\ \{(a,t)|\ t \in \mathbb{R}\}, & \text{for } a < 0 \text{ or } a > 1 \end{cases}$$

for Curtiss-Hirschfelder ODE

$$E\Omega = \begin{cases} \emptyset, & \text{for } a \leqslant -1 \\ \{(a,t)|\ -\arccos a + 2\pi k < t \leqslant \arccos a + 2\pi k\}, & \text{for } -1 < a < 1 \\ \{(a,t)|\ t \in \mathbb{R}\}, & \text{for } a \geqslant 1 \end{cases}.$$

Let $\Omega = \{(x,p,t) | x \geqslant a,\ p \in \mathbb{R},\ t \in \mathbb{R}\}$. Then for VanDerPol example

$$E\Omega = \begin{cases} \{(a,p,t)|\ p < 0,\ t \in \mathbb{R}\}, & \text{for } a \leqslant 0 \\ \{(a,p,t)|\ p \leqslant 0,\ t \in \mathbb{R}\}, & \text{for } a > 0 \end{cases}.$$

The next lemma says that each solution of ODE which starts inside Ω and reachs outside Ω crosses boundary $\partial\Omega$ in an exit point.

Lemma. Let $y_0 \in \Omega,\ s_{out} > 0,\ y(s_{out}; y_0) \notin \Omega$. Then exists $s_e \in [0, s_{out}]$ such that $y_e = y(s_e; y_0)$ is an exit point of ODE from Ω.

Proof. Let $s_e = \inf\{s \in [0, s_{out}] | y(s; y_0) \notin \Omega\}$. Let $y_e = y(s_e; y_0)$. If $s_e = 0$ then $y_e = y_0 \in \Omega$. If $s_e > 0$ then $y(s; y_0) \in \Omega$ for all $s \in [0, s_e[$. $y(s; y_0)$ as function of s is continuous,so $y_e \in \overline{\Omega}$.

The curve $y(s; y_e) = y(s_e + s; y_0)$ is a solution of ODE (2) within a range $[0, s_{out} - s_e]$ starting from y_e. Suppose that y_e is not an exit point. Then $\exists \delta > 0\ \forall s \in [0, \delta[\ y(s; y_e) \in \Omega$ and $\inf\{s \geqslant 0 | y(s; y_0) \notin \Omega\} \geqslant s_e + \delta$ that contradicts definition of s_e. $\blacksquare$.

Proposition. If exit hole of ODE from a set Ω is empty, then the set is positively invariant set.

Proof. Suppose that Ω is not positively invariant set. So there is a point $y_0 = (x_0, t_0) \in \Omega$, and $s > 0$ so that $y(s; y_0) \notin \Omega$. By previous lemma there is an exit point $y_e = y(s_e; y_0)$ for some s_e that contradicts to emptiness of $E\Omega$. $\blacksquare$.

3 Half spaces

We consider how to find exit holes of ODE from closed half space

$$\Omega = \{y | c^T y \geqslant a\}$$

with inward-pointing normal $c \in \mathbb{R}^{n+1}$, $c \neq 0$. Let y_0 is point on the boundary of Ω that is hyperplane $c^T y_0 = a$. The first-order expansion of general ODE solution near y_e is

$$y(s; y_0) = y_0 + F(y_0)s + o(s).$$

So

$$c^T y(s; y_0) = c^T y_0 + c^T F(y_0) + o(s) = a + c^T F(y_0) + o(s).$$

If $c^T F(y_0) > 0$ then y_0 is not exit point.
If $c^T F(y_0) < 0$ then y_0 is exit point.
If $c^T F(y_0) = 0$ then we can say nothing.

If F is continuously differentiable in the neighbourhood of y_e then there exists second order expansion of general ODE solution new y_e

$$y(s; y_0) = y_0 + F(y_0)s + \frac{\partial F(y_0)}{\partial y} F(y_0) \frac{s^2}{2} + o(s^2).$$

If $c^T F(y_0) = 0$ and $c^T \frac{\partial F(y_0)}{\partial y} F(y_0) > 0$ then y_0 is not exit point.
If $c^T F(y_0) = 0$ and $c^T \frac{\partial F(y_0)}{\partial y} F(y_0) < 0$ then y_0 is exit point.
If both $c^T F(y_0) = 0$ and $c^T \frac{\partial F(y_0)}{\partial y} F(y_0) = 0$ then we can say nothing.

Let us rewrite the above conditions in terms of $f(x,t)$ with $c = (c_x, c_t)$

$$c^T F(y) = c_x^T f(x,t) + c_t \tag{7}$$

$$c^T \frac{\partial F(y_0)}{\partial y} F(y_0) = c_x^T \left(\frac{\partial f(x,t)}{\partial x} f(x,t) + \frac{\partial f(x,t)}{\partial t}\right) \tag{8}$$

Example: Normal parallel to time Let normal is $c = (0, c_t)$. Then $c^T F(y) = c_t$ for any ODE. For positive half space $\Omega_+ = \{(x,t) | t \geqslant a,\ x \in \mathbb{R}^n\}$ exit hole is empty $E\Omega_+ = \emptyset$. For negative half space $\Omega_- = \{(x,t) | t \leqslant a,\ x \in \mathbb{R}^n\}$ exit hole is entire boundary $E\Omega_- = \partial\Omega_- = \{(x,a) |\ x \in \mathbb{R}^n\}$.

Scalar ODEs Now consider scalar ODE ($n = 1$) and $c_x \neq 0$. Let $k = -c_t/c_x$. The equation of half-plane whose boundary contains given point $y_0 = (x_0, t_0)$ is

$$c_x(x - x_0 - k(t - t_0)) \geqslant 0$$

The first-order condition is

$$c^T F(y) = c_x(f(x,t) - k).$$

When $c_x > 0$ the equation of half space is $x \geqslant x_0 + \underline{k}(t - t_0)$. If $f(x_0 + \underline{k}(t - t_0), t) > \underline{k}$ for $t \in [t_0, t_1]$ then this half-plane hasn't exit points with $t \in [t_0, t_1]$.
When $c_x < 0$ the equation of half space is $x \leqslant x_0 + \overline{k}(t - t_0)$. If $f(x_0 + \overline{k}(t - t_0), t) < \overline{k}$ for $t \in [t_0, t_1]$ then this half-plane hasn't exit points with $t \in [t_0, t_1]$.

Dahlquist ODE Let $x_0 > 0$.
The half-plane $x \leqslant x_0 + \overline{k}t$ hasn't exit points with $t \in [0, h)$ when $\forall t \in [0, h)$ $-x_0 - \overline{k}t < \overline{k}$ that is when $\overline{k} \geqslant -\frac{x_0}{1+h}$.
The half-plane $x \geqslant x_0 + \underline{k}t$ hasn't exit points with $t \in (0, h)$ when $\forall t \in (0, h)$ $-x_0 - \underline{k}t > \underline{k}$ for $t \in (0, h)$ that is when $\underline{k} \leqslant -x_0$. Second-order condition proves that there $(x_0, 0)$ isn't exit point too.

We have bounds

$$x_0(1 - t) \leqslant x(t; x_0) \leqslant x_0\left(1 - \frac{t}{1+h}\right)$$

$$x_0(1 - h) \leqslant x(h; x_0) \leqslant x_0 \frac{h}{1+h}.$$

For this ODE the lower bound is the same as forward-Euler point integration method returns, the upper bound is the same as backward-Euler point integration method returns.

When we bound $x(t; 1)$ by half-planes with $x_0 = 1$ the bounds are $1 - h \leqslant x(h; 1) \leqslant \frac{h}{1+h}$. The upper bound is asymptotically tight at $h \to +\infty$, the lower bound is not. However, we can try half-planes with $x_0 < 1$ for large h. The half-plane $x \geqslant 0 + 0t$ is better lower bound for $h > 1$.

So for each h the tightest enclosure of the solution of Dahtlqusit ODE $x(h;1)$ by two half-planes is

$$x(h;1) \in \begin{cases} [1-h, \frac{1}{1+h}], & \text{for } h \in [0,1] \\ [0, \frac{1}{1+h}], & \text{for } h \in [1,+\infty) \end{cases}.$$

In general the tightest lower bound is obtained by following semi-infinite programming problem (the problem for upper bound is similar)

$$\begin{gathered} b + \underline{k} \to \max \\ b \leqslant x_0 \\ f(b + \underline{k}t, t) \geqslant \underline{k},\ \forall t \in [t_0, t_1] \end{gathered}$$

We don't solve semi-infinite problem in non-linear case. Instead we enclose $f(x,t)$ by piece-wise linear bounds. In this case the semi-infinite problem becomes linear programming problem. The next section is about piece-wise linear enclosure by interval techniques.

4 Enclosure of function in a surface

In prevous section we used analitical calculations to prove that there are no exit holes in some part S of a surface. We checked that condition $c^T F(y) > 0$ is valid in S. In other words we checked that $m = \min_{y \in S} c^T F(y) > 0$.

Analytical calculations are possible for simple examples only. We use interval arithmetic to estimate minimum of a $c^T F(y)$ defined by an expression $c^T \mathsf{F}(y)$ in a set S.

The simplest way is natural interval evaluation. Let the interval hull of a set $\square S = \boldsymbol{s}$. Then $m \geqslant \inf \mathsf{F}(\boldsymbol{s})$. Sometimes the result of the calculation is tight enough.

Estimation of Autonomous scalar ODE. Let $F(y) = (f(x), 1)^T$ and $n = 1$. Let $f(x)$ is defined by such expression that $\mathsf{f}(\boldsymbol{x}) = \mathrm{range}_{x \in \boldsymbol{x}} f(x)$. Let $\square S = \boldsymbol{s} = \boldsymbol{x} \times [t_0, t_1]$. Then

$$\begin{gathered} \mathrm{range}_{y \in S} c^T F(y) = \mathrm{range}_{y=(x,t)^T \in S} (c_x f(x) + c_t) = \\ c_x \mathrm{range}_{x \in \boldsymbol{x}} f(x) + c_t = c_x \mathsf{f}(\boldsymbol{x}) + c_t = c^T \mathsf{F}(\boldsymbol{x}). \end{gathered}$$

If we use expression $\mathsf{f}(x) = -x$ for Dahlquist ODE and expression $\mathsf{f}(x) = \frac{1}{2} - \mathrm{abs}(x)$ for Nonsmooth ODE, then $\mathsf{f}(\boldsymbol{x}) = \mathrm{range}_{x \in \boldsymbol{x}} f(x)$ and natural interval evaluation gives us exact bound of m.

Estimation of Curtiss-Hirschfelder ODE. We use the expression $\mathsf{F}(y) = 50(\cos(t) - x)$. Let we have a surface

$$\begin{gathered} S = \{ y = (x,t)^T \mid c_x(x - x_0) + c_t(t - t_0) = 0, t \in [t_0, t_1] \} = \\ \{ y = (x,t)^T \mid x = x_0 + k(t - t_0), t \in [t_0, t_1] \} \end{gathered}$$

where $k = -\frac{c_t}{c_x}, x_1 = x_0 + k(t_1 - t_0), 0 \leqslant t_0 < t_1 \leqslant \frac{\pi}{2}$.

Suppose $k < 0$.

$$\begin{gathered}
\square S = \boldsymbol{s} = ([x_1, x_0], [t_0, t_1])^T \\
c^T \mathsf{F}(\boldsymbol{s}) = c_x \mathsf{f}([x_0, x_1], [t_0, t_1]) + c_t = \\
c_x 50(\cos[t_0, t_1] - [x_1, x_0]) + c_t = \\
c_x 50[\cos(t_1) - x_0, cos(t_0) - x_1] + c_t \supset \\
\text{range}_{t \in [t_0, t_1]} 50(cos(t) - x_0 - k(t - t_0)) = \text{range}_{y \in S} c^T F(y)
\end{gathered}$$

Simple interval extension doesn't give us exact estimation.

Mean value form We use a mean value form interval extension with the center $y^* \in S$ to obtain a better estimation than simple interval calculation.

$$c^T F(y) \in \boldsymbol{F}_{c,y^*}(y) = c^T \mathsf{F}(y^*) + \boldsymbol{l}^T (y - y^*), \tag{9}$$

where $\boldsymbol{l}^T = c^T \mathsf{F}'(\boldsymbol{s})$. Suppose that S is a convex hull of some number of boxes or points $S = \text{ch}\,(\bigcup_{i \in I} \boldsymbol{s}_i)$. The real functions $\underline{F}_{c,y^*}$ and $\overline{F}_{c,y^*}$ are respectively concave and convex functions of y. So they achive their minimum and maximum respectively at boxes - arguments of convex hull. Then

$$\text{range}_{y \in S} c^T F(y) \subseteq c^T \mathsf{F}(y^*) + \square \bigcup_{i \in I} \boldsymbol{l}^T (\boldsymbol{s}_i - y^*)$$

.

We want to choose $y^* \in S$ to obtain tighter lower bound. The choice will use only $\boldsymbol{l} = c^T \mathsf{F}'(\boldsymbol{s})$ and no other information about $F(y)$. We can choose $\underline{y}^* = \text{argmax}_{y^* \in \boldsymbol{s}} \min_{y \in \boldsymbol{s}} (y - y^*)^T \boldsymbol{l}$.

The formula for of y^* is provided by Baumann [12].

$$y^* = \begin{cases} \overline{s}_i, & \text{for } \underline{l}_i \leqslant 0 \\ \underline{s}_i, & \text{for } \overline{l}_i \geqslant 0 \\ (\bar{l}_i \underline{x}_i - \underline{l}_i \overline{x}_i)/(\bar{l}_i - \underline{l}_i), & \text{otherwise,} \end{cases}.$$

He showed that this value is optimal for estimation in $\boldsymbol{s}$. For all $y \in \boldsymbol{s}$

$$\inf \boldsymbol{F}_{c,y} \leqslant \boldsymbol{F}_{c,y^*}.$$

We choose for estimation in S by analogy

$$y^* = \operatorname*{argmax}_{y^* \in S} \min_{y \in S} (y - y^*)^T \boldsymbol{l}.$$

When the set is a convex hull of two boxes $S = \text{ch}\,(\boldsymbol{s}_0 \cup \boldsymbol{s}_1)$, the choice of y^* is a linear programming problem. It has $3n+1$ variables y_i^*, p_{0i}, $p1i$, α, m:

$$
\begin{gathered}
m \to \max \\
m \leqslant p_{01} + p_{02} + \cdots + p_{0n} \\
p_{0i} \leqslant \underline{\boldsymbol{l}}_i(\underline{\boldsymbol{s}}_{0i} - y_i^*),\ p_{0i} \leqslant \overline{\boldsymbol{l}}_i(\underline{\boldsymbol{s}}_{0i} - y_i^*),\ p_{0i} \leqslant \underline{\boldsymbol{l}}_i(\overline{\boldsymbol{s}}_{0i} - y_i^*),\ p_{0i} \leqslant \overline{\boldsymbol{l}}_i(\overline{\boldsymbol{s}}_{0i} - y_i^*) \\
m \leqslant p_{11} + p_{12} + \cdots + p_{1n} \\
p_{1i} \leqslant \underline{\boldsymbol{l}}_i(\underline{\boldsymbol{s}}_{1i} - y_i^*),\ p_{1i} \leqslant \overline{\boldsymbol{l}}_i(\underline{\boldsymbol{s}}_{1i} - y_i^*),\ p_{1i} \leqslant \underline{\boldsymbol{l}}_i(\overline{\boldsymbol{s}}_{1i} - y_i^*),\ p_{1i} \leqslant \overline{\boldsymbol{l}}_i(\overline{\boldsymbol{s}}_{1i} - y_i^*) \\
0 \leqslant \alpha \leqslant 1 \\
\alpha \underline{\boldsymbol{s}}_{0i} + (1-\alpha)\underline{\boldsymbol{s}}_{1i} \leqslant y_i^* \\
y_i^* \leqslant \alpha \overline{\boldsymbol{s}}_{0i} + (1-\alpha)\overline{\boldsymbol{s}}_{1i}
\end{gathered}
$$

These real equations emulate interval evaluation $\max_{y^* \in S} \min(\boldsymbol{l}(\boldsymbol{s}_0 - y^*) \cup \boldsymbol{l}(\boldsymbol{s}_1 - y^*))$

This estrimation is heuristic. The y^* may be suboptimal if S isn't a box. Nevertheless, the mean-value form remains correct for any $y^* in \boldsymbol{S}$. So we don't bother about rigorous solution of the LP problem. We use point LP solver like [14] and we don't bother about interval LP solver.

5 Trapezoid

Proposition Let S is the intersection of finite number of sets $S = \bigcap_{i \in I} S_i$. Then $ES = \bigcup_{i \in I}(ES_i \cap \overline{S})$.

In this paper we shall apply this proposition to a trapezoid of dimention $n+1$. It has two parallel facets which are boxes of dimension n. Parallel facets are ortogonal either to time axis or to one of state axis. We can consider this axis as integration variable. Such trapezoid is an intersection of $2(n+1)$ half spaces.

Let us write inequalities describing a trapezoid with integration variable t and with parallel facets $(\boldsymbol{x}_0, t_0)$ and $(\boldsymbol{x}_1, t_1)$.

$$
\begin{gathered}
S = S^L \cap \boldsymbol{S}^R \cap \bigcap_{1 \leqslant i \leqslant n} (S^{il} \cap S^{iu}) S^L = \{y = (x,t) \mid t \geqslant t_0\} \\
S^R = \{y = (x,t) \mid t \geqslant t_0\} \\
S^{il} = \{y = (x,t) \mid x_i \geqslant \underline{\boldsymbol{x}}_{0i} + \frac{\underline{\boldsymbol{x}}_{1i} - \underline{x}_{0i}}{t_1 - t_0}(t - t_0)\} \\
S^{ir} = \{y = (x,t) \mid x_i \geqslant \overline{\boldsymbol{x}}_{0i} + \frac{\overline{\boldsymbol{x}}_{1i} - \overline{x}_{0i}}{t_1 - t_0}(t - t_0)\} \\
F^L = S \cap \partial \boldsymbol{S}^L \\
F^R = S \cap \partial \boldsymbol{S}^R \\
F^{il} = S \cap \partial \boldsymbol{S}^{il} \\
F^{ir} = S \cap \partial \boldsymbol{S}^{ir}
\end{gathered}
$$

We want that exit hole of trapezoid is located only at the surface F^R. The proposition guarantees this if exit holes of other surfaces doesn't intersect with S. So we have a strong variant of Müller theorem. Let

$$f_i(x,t) > \frac{\underline{\boldsymbol{x}}_{1i} - \underline{\boldsymbol{x}}_{0i}}{t_1 - t_0}(t - t_0) \qquad \text{if } y = (x,t) \in F^{il} \tag{10}$$

$$f_i(y,t) < \frac{\overline{\boldsymbol{x}}_{1i} - \overline{\boldsymbol{x}}_{0i}}{t_1 - t_0}(t - t_0) \qquad \text{if } y = (x,t) \in F^{iu} \tag{11}$$

then $ES \subseteq F^R$. If we replace strong inequalities $>$ and $<$ by weak inequalities $\geqslant$ and $\leqslant$ the theorem is still correct, but its prove is more complicated.

The trapezoid with integration variable x_k is described in similar way. Müller theorem for such trapezoid will use $\frac{f_i(x,t)}{f_k(x,t)}$ instead of $f_i(x,t)$. The trapezoid will have a pair of side facets related to time. Müller theorem will use $\frac{1}{f_k(x,t)}$ for them.

Suppose that we want to build a trapezoid S with specified t_0 and t_1 which containes specified box $\boldsymbol{y}_{entry}$ such that $ES \subseteq F^R$. Also we want to obtain F^R as narrow as possible. We can
- guess apriory enclosures for each future surface $\boldsymbol{a}^{il}$ and $\boldsymbol{a}^{iu}$,
- evaluate $\boldsymbol{l}^{il} = \mathsf{f}'_i(\boldsymbol{a}^{il})$ and $\boldsymbol{l}^{iu} = \mathsf{f}'_i(\boldsymbol{a}^{iu})$,
- guess center points for each side facet $y^{il*} \in \boldsymbol{a}^{il}$ and $y^{ir*} \in \boldsymbol{a}^{ir}$,
- choose weights w_i of component width - and solve such a linear programming (written in macro style)

$$\sum_{i=1}^{i=n} w_i(\overline{x}_{1i} - \underline{x}_{1i}) \to \min$$

$$\min(\boldsymbol{l}^{il}(\boldsymbol{s}^{ilL} - y^{il*}), \boldsymbol{l}^{il}(\boldsymbol{s}^{ilR} - y^{il*})) \geq \frac{\underline{\boldsymbol{x}}_{1i} - \underline{x}_{0i}}{t_1 - t_0}(t - t_0)$$

$$\max(\boldsymbol{l}^{iu}(\boldsymbol{s}^{iuL} - y^{iu*}), \boldsymbol{l}^{iu}(\boldsymbol{s}^{iuR} - y^{iu*})) \geq \frac{\overline{\boldsymbol{x}}_{1i} - \overline{x}_{0i}}{t_1 - t_0}(t - t_0)$$

$$\boldsymbol{y}_{entry} \subseteq S$$

where $\boldsymbol{s}^{ilL} = F^{il} \cap F^L$, $\boldsymbol{s}^{iuL} = F^{iu} \cap F^L$, $\boldsymbol{s}^{ilR} = F^{il} \cap F^R$, $\boldsymbol{s}^{iuR} = F^{iu} \cap F^R$ are ridges of dimension $n-1$. The exact size of linear program depends on number n_J of non-zero items in system Jacobian
- $4n$ variables describing trapezoid $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$;
- $4n_J$ variables for partial interval products;
- $16n_J$ constraints simulating partial interval products;
- $4n$ constrains simulating min and max;
- $2n$ or $4n$ constraints to cover $\boldsymbol{y}_{entry}$;
- Total $4(n + n_J)$ variables and $8(n + 2n_J)$ constraints.

This procedure can be applied in iterative manner. In this case previous facets give us are blunted to get apriory enclosure. Also previous facests are used to get center points by a procedure described in the previous section.

6 Union of a sequence

Proposition Let S is the union of finite number of sets $S = \bigcup_{i=1}^{i=n} S_i$. Then $ES \subseteq \bigcup_{i \in I} ES_i$.

Proposition Let S is the union of finite number of sets $S = \bigcup_{i=1}^{i=n} S_i$. Let exit hole of each set is contained in the next sets $\forall 1 \leqslant i < n \; ES_i \subseteq \bigcup_{j=i+1}^{j=n} S_j$. Then $ES \subseteq ES_n$.

Corollary Let in previous proposition $T = \min_t (x,t) \in ES_n$. Then $S \cup \{(x,t) \mid \boldsymbol{t} \geqslant T, x \in \mathbb{R}^n\}$ is a positively invariant set.

These propositions explain how steps with different integration variables are glued together.

7 Experimental result

These schema was applied to simulation of Van der Pol example in time interval $[0, 500]$. The integration variable was alternatively u or v state variable. Integration variable t was used only at a few final steps to make results comparable with VNODELP solver.

Exit Hole: 448 steps took 18.17 sec
u=[-1.9086161385578748,-1.8226020274925632]
v=[0.0014443088246770168,0.0015699581152779204]

VNODELP ATol=3e-6: 79092 steps took 327.661 msec sec
u=[-1.8707838800728427,-1.8573010906442908]
v=[0.001495198885493725,0.0015178122071852266]

8 Conclusion

The Exit Hole method is a promising approach for verified solution of stiff IVPs. However, its simpest implementation (sequence of trapezoids) doesn't handle wrapping effect. Also its scalability with n is under the question.

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